Last time:

Vector valued functions

$$\boxed{\vec{r}(t) = \langle f(t), g(t), h(t) \rangle}$$

Vector (-valued) function: domain $\longrightarrow$ range

domain: set of real numbers

range: set of vectors

$$t \longmapsto \vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$
$$= \langle f(t), g(t), h(t) \rangle$$

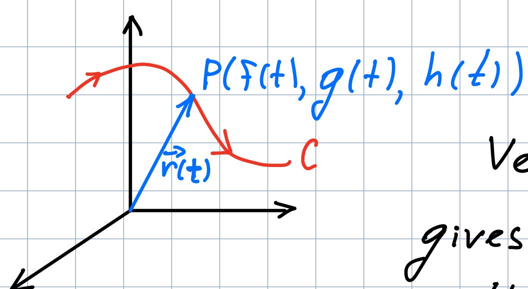
Rmk: domain = values t for which $\vec{r}(t)$ is defined.

Space curves

given functions

Set of points (x, y, z) with $\underbrace{x = f(t),\ y = g(t),\ z = h(t)}$, $t \in \underset{\text{interval}}{I}$ – "space curve" C.

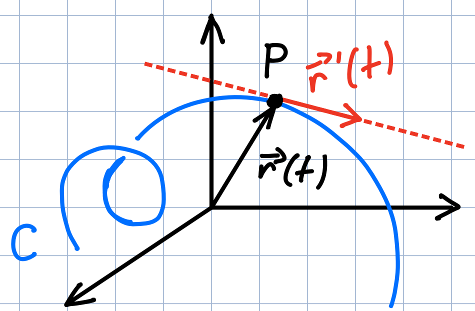
parametric equations of C, t - parameter



Vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ gives a "moving" vector, whose tip traces the curve C.

Today: Calculus of space curves

Derivatives: $\vec{r}(t)$ vector function $\rightsquigarrow$ $\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$

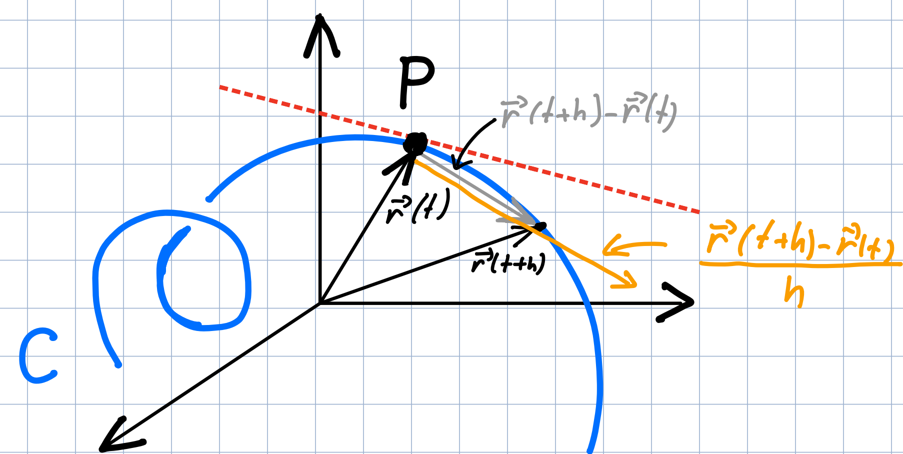


- $\vec{r}\,'(t)$ - tangent vector to curve C at point P (if $\vec{r}\,'(t) \neq 0$ and exists)
- tangent line to C at P - line through P, parallel to $\vec{r}\,'(t)$

- unit tangent vector: $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

* If $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

then $\vec{r}\,'(t) = f'(t)\,\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$



Ex: $\vec{r}(t) = (t+t^3)\vec{i} + \cos 2t\,\vec{j} + te^{-t}\vec{k}$

a) Find $\vec{r}\,'(t)$

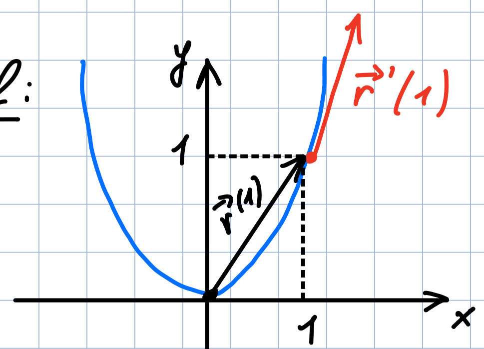
b) Find the unit tangent vector at the point with $t=0$

Sol: (a) $\vec{r}\,'(t) = (1+3t^2)\vec{i} - 2\sin 2t\,\vec{j} + (1-t)e^{-t}\vec{k}$

(b) $\vec{r}(0) = \vec{j}$, $\vec{r}\,'(0) = \vec{i} + \vec{k}$, $\vec{T}(0) = \dfrac{\vec{i}+\vec{k}}{|\vec{i}+\vec{k}|} = \dfrac{1}{\sqrt{2}}\vec{i} + \dfrac{1}{\sqrt{2}}\vec{k}$

Ex: On $\mathbb{R}^2$: $\vec{r}(t) = t\vec{i} + t^2\vec{j}$. Find $\vec{r}\,'(t)$, sketch $\vec{r}(1)$ and $\vec{r}\,'(1)$.

Sol:



$\vec{r}\,'(t) = \vec{i} + 2t\vec{j}$

$\Rightarrow \vec{r}(1) = \vec{i} + \vec{j}$, $\vec{r}\,'(1) = \vec{i} + 2\vec{j}$

Ex: C : $x = 2\cos t$, $y = \sin t$, $z = t$. Find eq. of the tangent line to C at point $(0, 1, \frac{\pi}{2})$

helix

corresponds to $t = \frac{\pi}{2}$

Sol: $\vec{r}(t) = \langle 2\cos t, \sin t, t \rangle$

$\vec{r}\,'(t) = \langle -2\sin t, \cos t, 1 \rangle \Rightarrow \vec{r}\,'(\frac{\pi}{2}) = \langle -2, 0, 1 \rangle$

The tangent line passes through $(0, 1, \frac{\pi}{2})$ and parallel to $\langle -2, 0, 1 \rangle$

$x = 0 - 2t = -2t$

$y = 1 + 0t = 1$

$z = \frac{\pi}{2} + t$

Second derivative: $\vec{r}\,'' = (\vec{r}\,'(t))'$

Ex: $\vec{r}(t) = \langle 2\cos t, \sin t, t \rangle \Rightarrow \vec{r}\,'(t) = \langle -2\sin t, \cos t, 1 \rangle$

$\Rightarrow \vec{r}\,''(t) = \langle -2\cos t, -\sin t, 0 \rangle$

Differentiation rules

$(\vec{u}(t) + \vec{v}(t))' = \vec{u}\,'(t) + \vec{v}\,'(t)$

$(c\,\vec{u}(t))' = c\,\vec{u}\,'(t)$

$(\vec{u}(f(t)))' = f'(t)\,\vec{u}\,'(f(t))$

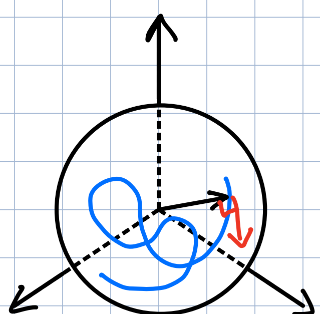
↑ chain rule

$(f(t)\,\vec{u}(t))' = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}\,'(t)$

$(\vec{u}(t) \cdot \vec{v}(t))' = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$

$(\vec{u}(t) \times \vec{v}(t))' = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$

Ex: If $|\vec{r}(t)| = c$ - constant, then $\vec{r}\,'(t)$ is orthogonal to $\vec{r}(t)$ for all t.



Indeed, $\frac{d}{dt}\Big(\underbrace{\vec{r}(t) \cdot \vec{r}(t)}_{c^2}\Big) = \vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t)$

$0 \;/\!/ \qquad\qquad = 2\,\vec{r}\,'(t) \cdot \vec{r}(t)$

Integrals:

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}$$

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

Fund. Thm of Calculus (in vector setting)

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big|_a^b = \vec{R}(b) - \vec{R}(a)$$

$$\vec{R}'(t) = \vec{r}(t)$$

We denote $\vec{R}(t) = \int \vec{r}(t)\,dt$ (indefinite integral)

Ex: $\vec{r}(t) = 2\cos t\,\vec{i} + \sin t\,\vec{j} + 2t\,\vec{k}$

$$\int \vec{r}(t)\,dt = 2\sin t\,\vec{i} - \cos t\,\vec{j} + t^2\vec{k} + \vec{C}$$

$$\int_0^{\frac{\pi}{2}} \vec{r}(t)\,dt = \left|2\sin t\,\vec{i} - \cos t\,\vec{j} + t^2\vec{k}\right|_0^{\frac{\pi}{2}} = 2\vec{i} + \vec{j} + \frac{\pi^2}{4}\vec{k}$$